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SOME PROPERTIES OF THE GENERALIZED FUNCTION WEIGHTED METRIC SPACES

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Abstract. *This paper introduces a function weighted b-metric space ($\mathfrak{F}$ b-metric space) as a generalization of the function weighted metric (F-metric space). We also propose and prove some properties of the $\mathfrak{F}$ b-metric space for logarithmic-like function.*

Keywords. *logarithmic-like function , b-metric space, function weighted metric, function weighted b- metric.*

Abstrak. Artikel ini memperkenalkan ruang b-metrik terboboti fungsi (ruang b-metrik $\mathfrak{F}$) sebagai suatu perumuman dari ruang metrik terboboti fungsi (ruang F-metrik).Artikel ini mengajukan dan membuktikan beberapa sifat ruang b-metrik $\mathfrak{F}$ untuk fungsi mirip logaritma.

Kata Kunci. Fungsi mirip logaritma, ruang b-metrik, metrik terboboti fungsi, b-metrik terboboti fungsi.

1. INTRODUCTION

The b-space metric was introduced by Bakhtin in 1989 [1] and used by Czerwik in 1993 for the fixed point of contracting Banach [2]. Dung in 2019 made the b-metric space metrizable on a certain metric space [3]. George [2015] introduced the rectangular b-metric space which is a generalization of the rectangular metric space [4]. The rectangular b-metric space has been introduced by Branciari (2000) on the existence of points in the Banach contract function and the Kannan contraction type [5]. Khamsi (2010) has generalized a rectangular metric space that replaces the conditions of rectangular inequalities into inequalities for n-angular [6]. One of the generalizations of the other b-metric spaces is the s-relaxed b-metric space [7]. In 2018, Jleli and Samet introduced a generalization of the metric space called the function weighted metric (F-metric) space [8].

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The main goal of this paper is to introduce the function weighted b- metric (the F b-metric space) which is a generalization of the F–metric space and reveal several properties related to the properties of the F b-metric.

2. PRELIMINARIES

In order to work out the main results, the following are some of the definitions and examples needed.

Definition 1. (see[1,2]) Let X be a non-empty set and a real number $b \geq 1$. Let $d: X \times X \rightarrow [0, \infty)$ be a function, then (X, d) is called *b-metric space* if the following conditions are satisfied

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq k(d(x, z) + d(z, y))$

for all $x, y, z \in X$.

Definition 2. (see [8]) Let $f: (0, +\infty) \rightarrow \mathbb{R}$ be a function, non-decreasing, f is *called logarithmic-like* if it satisfies

$$\lim_{t \rightarrow 0} f(t) = -\infty.$$

We denote

$$F = \{ f: (0, +\infty) \rightarrow \mathbb{R} \mid f \text{ non} - \text{decreasing, and logarithmic} - \text{like} \}.$$

So we have if $f \in F$, then for any $r > 0$ and $K \geq 0$, there is $\delta > 0$, so that for every $0 < s < \delta$, it holds $f(s) < f(r) - K$.

Example 1. (see [8]) Some examples for logarithmic-like function are $f(x) = \ln x$, $f(x) = x - \frac{1}{x}$, and $f(x) = x - e^{\frac{1}{x}}$, for $x \in (0, +\infty)$.

Definition 3. (see [8]). Let X be a non-empty set, $f \in F$, $0 \leq K < +\infty$. A mapping $\rho: X \times X \rightarrow [0, +\infty)$ is called a *function weighted metric* (F *−metric*), if for all $x, y \in X$, ρ satisfies the following conditions:

A1. $\rho(x, y) = 0$, if and only if $x = y$,

A2. $\rho(x, y) = \rho(y, x)$,

A3. $\rho(x, y) > 0$ then $f\big(\rho(x, y)\big) \leq f(\sum_{j=0}^{N-1} \rho\big(a_j, a_{j+1}\big) + K$, for every $\{a_0 = x,\ a_1,\ a_2, \dots, a_N = y\} \subset X$ and $N \in \mathbb{N}, N \geq 2$. The pair (X, ρ) is called a *function weighted metric space* (*F-metric space*) .

In the following, we define a function weighted b-metric which is a generalization of the function weighted metric, as follows:

Definition 4. Let X be a non-empty set, $f \in F,\ 0 \leq K < +\infty,\ b \geq 1$. A mapping $\rho: X \times X \to [0, +\infty)$ is called a function weighted b-metric ($F\ b-metric$) , if for all $x, y \in X$, ρ satisfies the following conditions:

B1. $\rho(x, y) = 0,$ if and only if $x = y,$

B2. $\rho(x, y) = \rho(y, x),$

B3. $\rho(x, y) > 0$ then $f\big(\rho(x, y)\big) \leq f(\sum_{j=0}^{N-1} b^j \rho\big(a_j, a_{j+1}\big) + K,$

for every $\{a_0 = x,\ a_1,\ a_2, \dots, a_N = y\} \subset X$ and $N \in \mathbb{N}, N \geq 2.$

The pair (X, ρ) is called a *function weighted b-metric space* ($F\ b-metric$ *space*).

Remarks. If $b = 1$, then the above definition is the definition of the function weighted metric.

However, the function weighted b-metric becomes b-metric if $f(x) = \ln x$, $K = 0,$ and $N = 2$. Also the b-metric space is not necessarily a function of the weighted metric space. This can be shown in the following Examples.

Example 2. Let $X = R$ be a set of all real numbers. Define a function $\rho(x, y) = |x - y|^p$ with $p \geq 2$ and $x, y \in X$, then ρ is a b-metric with $b = 2^{p-1}$. However, ρ is not a function weighted metric, this can be shown as follows. Suppose ρ is a function weighted metric. It means that there is a function $f \in F$ that satisfies the of axiom A3 of the definition of function weighted metric. We choose $t_j \in X$ and take any $m \in \mathbb{N}$. Define $t_j = \frac{2j}{m}$ for $j = 0,1,2, \dots., m - 1$. From the of axiom A3, we have

$$f(\rho(0,2)) \leq f(\sum_{j=0}^{m-1} \left|t_j - t_{j+1}\right|^p) + K = f(\sum_{j=0}^{m-1} \left|\frac{2j}{m} - \frac{2(j+1)}{m}\right|^p) + K$$

$$= f(\frac{1}{m^p} \sum_{j=0}^{m-1} |2j - 2(j+1)|^p) + K = f(\frac{2^p}{m^{p-1}}) + K. \tag{1}$$

So, from (1) we get

$$f(4) = f\big(\rho(0,2)\big) \leq f\left(\frac{2^p}{m^{p-1}}\right) + K. \tag{2}$$

Since f has a *logarithm-like property*, then we have $\frac{2^p}{m^{p-1}} \to 0$, as $m \to +\infty$ and consequently from (2) we have $f(\frac{2^p}{m^{p-1}}) + K \to -\infty$, as $m \to +\infty$, which is a contradiction, because $f\left(\frac{2^p}{m^{p-1}}\right) \geq f(4) - K$. So, ρ is not a function weighted metric. However, ρ is a function weighted *b*-metric. It is shown as follows. To prove condition B3, we use a property

$$|x-y|^p \leq 2^{p-1}(|x-a|^p + |a-y|^p).$$

Let $N \in \mathbb{N}, N \geq 2$, and a set $\{a_1 = x,\ a_2,\ a_3, \dots, a_N = y\} \subset X$. Then we have

$$\rho(x,y) = |x-y|^p$$

$$= e^{\alpha}|a_1 - a_2 + a_2 - a_3 + a_3 - a_4 + a_4 - \cdots a_{N-1} - a_N|^p$$

$$\leq (2^{p-1}|a_1 - a_2|^p + (2^{p-1})^2|\, a_2 - a_3|^p + (2^{p-1})^3|\, a_3 - a_4|^p + \cdots$$
$$+ (2^{p-1})^{N-1}|\, a_{N-1} - a_N|^{p-1})$$

$$= \sum_{j=1}^{N-1} (2^{p-1})^j \left|a_j - a_{j+1}\right|^p = \sum_{j=1}^{N-1} (2^{p-1})^j\, \rho\big(a_j,\ a_{j+1}\big).$$

So, we obtain

$$\rho(x,y) \leq \sum_{j=1}^{N-1} (2^{p-1})^j\, \rho\big(a_j,\ a_{j+1}\big).$$

Thus ρ is a function weighted *b*-metric with $f(x) = \ln x$, $b = 2^{p-1}$ and $K = 0$.

Example 3. Let $X = [0,1]$, define a function $\rho(x,y) = |x-y|^2 e^{|x-y|}$ for all $x, y \in X$. To prove condition B3, we use a property

$$|x-y|^2 \leq 2(|x-a|^2 + |a-y|^2).$$

Let $N \in \mathbb{N}, N \geq 2$, and a set $\{a_1 = x,\ a_2,\ a_3, \dots, a_N = y\} \subset X$. Then we have

$$\rho(x,y) = |x-y|^2 e^{|x-y|}$$

$$= |a_1 - a_2 + a_2 - a_3 + \cdots + a_{N-1} - a_N|^2 e^{|a_1-a_2+a_2-a_3+\cdots+a_{N-1}-a_N|}$$

$$\leq (2|a_1 - a_2|^2 + (2)^2|a_2 - a_3|^2 + \cdots + (2)^{N-1}|a_{N-1} - a_N|^2) e^{|a_1-a_2+a_2-a_3+\cdots+a_{N-1}-a_N|}$$

$$\leq \big(2|a_1 - a_2|^2 e^{|a_1-a_2|+|a_2-a_3+a_{N-1}-a_N|} + (2)^2|a_2 - a_3|^2 e^{|a_2-a_3|+|a_1-a_2+a_3-a_N|} + \cdots + (2)^{N-1}|a_{N-1} - a_N|^2\big) e^{|a_{N-1}-a_N|+|a_1-a_{N-1}|}$$

$$= \big(2|a_1 - a_2|^2 e^{|a_1-a_2|+|a_2-a_N|} + (2)^2|a_2 - a_3|^2 e^{|a_2-a_3|+|a_1-a_2+\cdots+a_{N-1}-a_N|} + \cdots + (2)^{N-1}|a_{N-1} - a_N|^2 e^{|a_{N-1}-a_N|+|a_1-a_2+a_{N-1}-a_N|}\big)$$

Since $a_0 = x,\ a_2,\ a_3, \dots, a_N = y \in [0,1]$, then we have

$$\rho(x,y) = |x-y|^2 e^{|x-y|}$$

$$\leq 2|a_1 - a_2|^2 e^{|a_1-a_2|} e^2 + (2)^2 |a_2 - a_3|^2 e^{|a_2-a_3|} e^2 + \cdots + (2)^{N-1}|a_{N-1} - a_N|^2 e^2$$

$$= e^2\big(2|a_1 - a_2|^2 e^{|a_1-a_2|} + (2)^2|a_2 - a_3|^2 e^{|a_2-a_3|} + \cdots + (2)^{N-1}|a_{N-1} - a_N|^2\big)$$

$$= e^2\left(\sum_{j=1}^{N-1} (2)^j \left|a_j - a_{j+1}\right|^2 e^{|a_j-a_{j+1}|}\right)$$

$$= e^2\left(\sum_{j=1}^{N-1} (2)^j \rho\big(a_j,\ a_{j+1}\big)\right). \qquad (3)$$

So, from (3) we obtain

$$\rho(x,y) \leq e^2 \sum_{j=1}^{N-1} (2^{p-1})^j \rho\big(a_j,\ a_{j+1}\big).$$

So, we have

$$\ln \rho(x,y) \le 2 + \ln \sum_{j=1}^{N-1} (2^{p-1})^j \rho(a_j,\ a_{j+1}).$$

This is a function weighted b-metric with the function $f(x) = ln\, x$, $b = 2^{p-1}$ and $K = 2$.

From Definition 2, replace A3 with the of A3* as follows: there is $\gamma > 0$ such that if $\rho^*(x,y) > 0$ holds,

$$\rho^*(x,y) \le \gamma \sum_{j=1}^{N-1} \rho^*(a_j, a_{j+1}),\ a_1 = x, \qquad a_N = y.$$

Where ρ^* is known as a *s-relaxed metric* [8]. It is clear that ρ^* satisfies B3 with $f(x) = \ln x,\ x > 0,\ b = 1$ and $K = \ln \gamma$. So, s-relaxed metric is a function weighted b- metric.

Example 4. Let X=[1,2], define the function

$$\rho(x,y) = \begin{cases} 0, & \text{If } x = y \\ 2^{|x-y|}, & \text{if } x \ne y \end{cases}$$

for all $x, y \in X$. For conditions B1 and B2, it is clearly satisfied. To prove condition B3, choose a function $f(t) = t + 1 - \frac{1}{t}$ for any $t \in (0, +\infty)$, it is clear that $f \in \mathfrak{F}$.

Let (a_n) be any finite sequence in $X = [1,2]$, where $(a_1, a_N) = (x, y)$ for all $x, y \in X$, and $\rho(x,y) > 0$, for $n = 1,2,3, \dots, N$. So, we have

$$3 + f\left(\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})\right) - f(\rho(x,y))$$

$$= 3 + 1 + \sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1}) - \frac{1}{\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})} - \rho(x,y) - 1 + \frac{1}{\rho(x,y)}$$

$$= 3 + \sum_{j=1}^{N-1} b^j 2^{|a_j - a_{j+1}|} - \frac{1}{\sum_{j=1}^{N-1} b^j 2^{|a_j - a_{j+1}|}} - 2^{|x-y|} + \frac{1}{2^{|x-y|}}$$

$$= 3 - \frac{1}{\sum_{j=1}^{N-1} b^j 2^{|a_j - a_{j+1}|}} - 2^{|x-y|}. \tag{4}$$

Since $b \ge 1$ and $x, y \in [1,2]$, then from (4) we obtain

$$3 + f\left(\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})\right) - f(\rho(x,y)) \geq 3 - \frac{1}{\sum_{j=1}^{N-1} b^j 2^{|a_j - a_{j+1}|}} - 2^{|x-y|}$$

$$\geq 2 - 1 - 2 = 0. \quad (5)$$

So, from (5) we get

$$f(\rho(x,y)) \leq f\left(\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1})\right) + 3.$$

Thus, ρ is a function weighted b-metric with $b \geq 1$, $f(t) = t + 1 - \frac{1}{t}$, $t > 0$, and $K = 3$.

Definition 5. Let (X, ρ) be a function weighted b-metric ($F\ b - metric$) and $p \in X$, then $N_r(p) = \{x \in X \mid \rho(x,\ p) < r\}$ is called an open neighborhood of p ($F\ b -$ *open neighborhood* of p). $G \subset X$ is called $F\ b -$ *open* set in X, if for any $y \in G$, there is $N_r(y)$, such that $N_r(y) \subset G$. Furthermore, if K is a $F\ b -$ open in X, then K^c is called $F\ b$ - *closed* in X.

3. MAIN RESULTS

Theorem 1. *Let* $(f, K) \in F \times [0, +\infty)$ *and* (X, ρ) *be a function weighted b-metric space (*$F\ b - metric\ space$*) and* $\{a_n\}$ *be a sequence in X. If* $\rho(a_n, a) \to 0$, *as* $n \to \infty$, *and for any G of* $F\ b -$ *open set in X containing* a, *there is a positive integer* N, *such that for any* $n \geq N$, *then* $a_n \in G$.

Proof. Since $a \in G$ and G open in $F\ b - metric\ space\ X$, then there is a open neighborhood $N_r(a)$ such that $N_r(a) \subset G$. Since $\rho(a_n, a) \to 0$, as $n \to \infty$, then there is a positive integer N, such that for any $n \geq N$, we have $\rho(a_n, a) < \frac{1}{2nb}$. Let $N_{\frac{1}{2nb}}(a_n)$ be an open neighborhood of a_n in X. We will show that $N_{\frac{1}{2nb}}(a_n) \subset N_r(a)$. Taking any $x \in N_{\frac{1}{2nb}}(a_n)$, $x \neq a$, we have $\rho(x, a) > 0$, then by using of B3, we obtain

$$f(\rho(x,a)) \leq f\left(b\big(\rho(x, a_n) + \rho(a_n, a)\big)\right) + K$$

$$\leq f\left(b\left(\frac{1}{2nb} + \frac{1}{2nb}\right)\right) + K = f\left(\frac{1}{n}\right) + K.$$

Since $f \in F$ and $r > 0$, then there is $\sigma > 0$ such that for any $0 < t < \sigma$ the following holds

$$f(t) < f(r) - K.$$

For the next, we choose a positive integer N, such that for any $n \geq N, \frac{1}{n} < \sigma$, then we get

$$f\left(\frac{1}{n}\right) < f(r) - K.$$

So, we get

$$f\big(\rho(x,a)\big) < f(r).$$

Since non decreasing monotonic, we obtain $\rho(x,a) < r$. This means $x \in N_r(a)$. So, we get that $N_{\frac{1}{2nb}}(a_n) \subset N_r(a) \subset G$ for any $n \geq N$. So it is proved that for any $n \geq N$, then $a_n \in$ G.

Theorem 2. *Let* $(f,K) \in F \times [0,+\infty)$ *and* (X,ρ) *be a function weighted b-metric space (*$F\ b - metric\ space$*). Suppose* $V_r(p) = \{x \in X |\ \rho(x,p) \leq r\}$*, and for any sequence* $(a_n) \subset X$ *holds a property: For any* $\tau > 0$*, there exists a positive integer N such that* $\rho(a_N,p) < \tau$*. Then* $V_r(p)$ is a $F\ b$ *−closed in* X*.*

Proof. Let (a_n) be a sequence in $V_r(p)$ that converges ($Fb - convergent$) to $a \in X$. We show that $a \in V_r(p)$. We have $(a_n) \subset V_r(p)$, this means $\rho(a_n,p) \leq r$. Since $r > 0$, then there is $\delta > 0$ such that for any $0 < s < \delta$, it holds $f(s) < f(r) - K$. Likewise, it is true that $\rho(a_N,p) < \frac{\delta}{3b}$ for a non-negative integer N. Since $\lim_{n\to+\infty} \rho(a_n,a) = 0$, then we have $\rho(a_n,a) < \frac{\delta}{3b} < \delta$ for any $n \geq N$. Since $\rho(a,p) > 0$, then by using of B3, then we have

$$f\big(\rho(a,p)\big) \leq f\Big(b\big(\rho(a,a_N) + \rho(a_N,p)\big)\Big) + K < f\left(b\left(\frac{\delta}{3b} + \frac{\delta}{3b}\right)\right) + K$$

$$= f\left(\frac{\delta}{3} + \frac{\delta}{3}\right) + K = f\left(\frac{2\delta}{3}\right) + K < f(r) - K + K = f(r).$$

So, we get $\rho(a,p) \leq r$, this means that $a \in V_r(p)$. Therefore, $V_r(p)\ F\ b -$ closed in X.

Theorem 3: *Let* $(f,K) \in F \times [0,+\infty)$ *and* (X,ρ) *be a function weighted b-metric space (*$F\ b - metric\ space$*). Supose* (a_n) *is a sequence in* X *which satisfies:*

$$\rho(a_n,\ a_{n+1}) \leq \frac{c}{b}\rho(a_{n-1},\ a_n), \tag{6}$$

where $0 < c < 1$. *Then* (a_n) *is Cauchy sequence in* F *b—metric space X.*

Proof. By using iteration of (6), we obtained

$$\rho(a_n, a_{n+1}) \leq \frac{c^n}{b^n}\rho(a_0, a_1).$$

So, we have

$$b^n \rho(a_n, a_{n+1}) \leq c^n \rho(a_0, a_1). \tag{7}$$

Let $m > n$, then from (7) we get

$$\sum_{i=n+1}^{m} b^i \rho(a_i, a_{i+1}) \leq \sum_{i=n+1}^{m} c^i \rho(a_0, a_1) \leq \frac{c^n}{1-c}\rho(a_0, a_1). \tag{8}$$

Since $0 < c < 1$, then from (8), if $n \to +\infty$, then $\frac{c^n}{1-c}\rho(a_0, a_1) \to 0$. This means, that for any $\gamma > 0$, there is $N \in \mathbb{N}$ such that for any $n \geq N$ we have

$$0 < \frac{c^n}{1-c}\rho(a_0, a_1) < \gamma. \tag{9}$$

Since $(f, K) \in F \times [0, +\infty)$, That means f is a non-decreasing and logarithmic-like function. So, for every $\varepsilon > 0$ there exists $\gamma > 0$ such that for any $s \in (0, \gamma)$ we have $f(s) < f(\varepsilon) - K$. Therefore, from (8) and (9), and for $m > n \geq N$, we have

$$f\left(\sum_{i=n+1}^{m} b^i \rho(a_i, a_{i+1})\right) \leq f\left(\frac{c^n}{1-c}\rho(a_0, a_1)\right) < f(\varepsilon) - K. \tag{10}$$

By using of B3, and (10) we obtain

$$\rho(a_m, a_n) > 0 \text{ maka } f\big(\rho(a_m, a_n)\big) \leq f(\sum_{i=n+1}^{m} b^i \rho(a_i, a_{i+1}) + K < f(\varepsilon).$$

Since f is a non-decreasing function, then $\rho(a_m, a_n) < \varepsilon$ for any $m > n \geq N$. This show that (a_n) is a Cauchy sequence in F b –metric space X.

Theorem 4. *Let* $(f, K) \in F \times [0, +\infty)$ *and* (X, ρ) *be a complete function weighted b-metric space (complete* F $b-metric\ space$*). Suppose* $W_r(a_0) = \{x \in X \mid \rho(x, a_0) \leq r\}$ *and* $a_1 \in W_r(a_0) \backslash \{a_0\}$ *with* $\rho(a_0, a_1) < \frac{r(1-c)}{2b^2(r+1)}$, *where* $0 < c < 1$. *If* (a_n) *is a sequence in* X, *that satisfies (6) and* $f\left(\frac{r}{b(r+1)}\right) < f(r) - K$, *then* (a_n)*is* F *b - convergent in* $W_r(a_0)$.

Proof. Since $a_0 \in W_r(a_0)$ and $a_1 \in W_r(a_0)\setminus\{a_0\}$, then from (7) we have

$$\rho(a_1, a_2) \leq \frac{c}{b}\rho(a_0, a_1) < \frac{c(1-c)r}{2b^3(r+1)}. \quad (11)$$

Since $\rho(a_0, a_2) > 0$, and using B3 and (11), we have

$$\begin{aligned} f(\rho(a_0, a_2)) &\leq f\left(b\left(\rho(a_0, a_1) + \rho(a_1, a_2)\right)\right) + K \\ &\leq f\left(b\left(\frac{r(1-c)}{2b^2(r+1)} + \frac{c(1-c)r}{2b^3(r+1)}\right)\right) + K \\ &= f\left(\frac{r(1-c)}{2b(r+1)} + \frac{cr(1-c)}{2(r+1)b^2}\right) + K \\ &\leq f\left(\frac{r}{2b(r+1)} + \frac{r}{2(r+1)b}\right) + K \\ &= f\left(\frac{r}{b(r+1)}\right) + K \\ &< f(r) - K + K = f(r). \end{aligned}$$

Since f is a non-decreasing function, then $\rho(a_0, a_2) < r$. So, we have $a_2 \in W_r(a_0)$. In the same way, we also have

$$\rho(a_2, a_3) \leq \frac{c}{b}\rho(a_1, a_2) \leq \left(\frac{c}{b}\right)^2 \rho(a_0, a_1) = \left(\frac{c}{b}\right)^2 \frac{r(1-c)}{2b^2(r+1)}.$$

So, we get

$$\begin{aligned} f(\rho(a_0, a_3)) &\leq f\left(b\rho(a_0, a_1) + b^2\rho(a_1, a_2) + b^2\rho(a_2, a_3)\right) + K \\ &\leq f\left(\left(\frac{r(1-c)}{2b(r+1)} + \frac{cr(1-c)}{2b(r+1)} + \frac{c^2r(1-c)}{2b(r+1)}\right)\right) + K \\ &= f\left(\frac{r(1-c)}{2b(r+1)}(1 + c + c^2)\right) + K \\ &= f\left(\frac{r(1-c)}{2b(r+1)}\frac{(1-c^3)}{1-c}\right) + K \\ &= f\left(\frac{r(1-c^3)}{b(r+1)}\right) + K \\ &\leq f\left(\frac{r}{b(r+1)}\right) + K \\ &< f(r) - K + K = f(r). \end{aligned}$$

So, we have

$$f\big(\rho(a_0,a_3)\big) \le f(r).$$

It concludes that $a_3 \in W_r(a_0)$.

Thus, in general for $n = 1,2,3,...$, we have

$$\rho(a_{n-1},a_n) \le \left(\frac{c}{b}\right)^{n-1} \frac{r(1-c)}{2b^2(r+1)}$$

and

$$f\big(\rho(a_0,a_n)\big) \le f\left(\frac{r(1-c^n)}{b(r+1)}\right) + K \le f\left(\frac{r}{b(r+1)}\right) + K$$

$$< f(r) - K + K = f(r). \tag{12}$$

From (12), it is obtained $\rho(a_0,a_n) < r$. Thus, we have

$$a_n \in W_r(a_0), \text{ for every } n = 1,2,3,..... \tag{13}$$

Since by using Theorem 3 and (13), we have that (a_n), is a Cauchy sequence in $W_r(a_0)$. Then, by using Theorem 2, we have $W_r(a_0)$ is closed. Since X is complete, then (a_n) is convergent in F b –metric space X. Since $W_r(a_0)$ is a closed, and $a_n \in W_r(a_0)$, then (a_n) is a F b -convergent in $W_r(a_0)$.

Theorem 5. *Suppose* $(f,K) \in F \times [0,+\infty)$ *and let* (X,ρ) *be a complete function weighted b-metric space* (F $b-metric\ space$). *If* (a_n) *is a convergent sequence in* F *b- metric space* X, *then the limit of* (a_n) *is unique.*

Proof. Suppose $\lim_{n\to+\infty} \rho(a_n,a^*) = 0$, $\lim_{n\to+\infty} \rho(a_n,s^*) = 0$, and $a^* \ne s^*$. Since $\rho(a^*,s^*) > 0$, then from of B3, we have

$$f\big(\rho(a^*,s^*)\big) \le f\Big(b\big(\rho(a^*,a_n) + \rho(a_n,s^*)\big)\Big) + K. \tag{14}$$

Since $\lim_{n\to+\infty} \rho(a_n,a^*) = 0$ and $\lim_{n\to+\infty} \rho(a_n,s^*) = 0$, then we have

$$b\big(\rho(a^*,a_n) + \rho(a_n,s^*)\big) \to 0, \text{ as } n \to +\infty.$$

Then from (14) and by using the logarithmic-like property of f, we get

$$\lim_{n\to\infty} f\big(c\rho(a^*,a_n) + b\rho(a_n,s^*)\big) + K = -\infty,$$

which is a contradiction.

Theorem 6. *Suppose* $(f,K) \in F \times [0,+\infty)$ *and let* (X,ρ) *be a function weighted b-metric space. If* $lim_{t\to+\infty} f(t) = +\infty$ *, then for any* $t > 0$ *there exists* $M > t$ *such that* $f(t) < f(M) - K$.

Proof. Since $f \in F$ and $\lim_{t\to+\infty} f(t) = +\infty$, we have f no-deceasing, logarithmic –like, and upper unbounded. Suppose there exists $M > 0$ such that for any $s > M$, it holds $f(M) \geq f(s) - K$. So, we have $f(s) \leq f(M) + K$. Which is a contradiction, because f is not upper bounded.

Theorem 7. *Suppose* $(f,K) \in F \times [0,+\infty)$ *and let* (X,ρ) *be a function weighted b-metric space. If* $lim_{t\to+\infty} f(t) = +\infty$ *and* $(a_n) \subset X$ *is a Cauchy sequence in F b—metric space , then* (a_n) *is bounded in* X.

Proof. Suppose (a_n) is a Cauchy sequence, then there is a positive integer N, such that $\rho(a_n, a_N) < \frac{1}{b}$ for any $n \geq N$. Let $c \in \Re$ and

$$L = maks\left\{b\rho(a_1, a_N), b\rho(a_2, a_N), \dots, b\rho(a_{N-1}, a_N), \frac{1}{b}\right\}.$$

For $\rho(a_n, c) > 0$, then using B3, we get

$$f(\rho(a_n,c)) \leq f\left(b(\rho(a_n,a_N) + \rho(a_N,c))\right) + K$$
$$\leq f\left(b(L + \rho(a_N,c))\right) + K. \quad (15)$$

From Theorem 6, and since $\lim_{t\to+\infty} f(t) = +\infty$ and f non-decreasing, then there is $M > L + b\rho(a_N, c) > 0$ such that $f\left((L + b\rho(a_N,c))\right) < f(M) - K$. So, from (15) we have

$$f(\rho(a_n,c)) \leq f\left((L + b\rho(a_N,c))\right) + K < f(M).$$

Since f is a non-decreasing, we get $\rho(a_n, c) < M$, for every $n = 1,2,3, \dots$. Thus (a_n) is bounded in X.

Next, we will show that (X,ρ)) which is a weighted *b*-metric function, is also a metrizable space, this is shown in the following theorem.

Theorem 8. *Suppose* $(f,K) \in F \times [0,+\infty)$ *and let* (X,ρ) *be a function weighted b-metric space. If* $d(x,y) = inf_{\ N\in\mathbb{N}}\{\sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1}) \mid (a_n) \subset X, a_1 =$

$x, a_N = y\}$, $N \geq 2$, *then* d *is a metric on* X. *Furthermore,* (a_n) *converges to a in* (X, ρ) *if and only if* (a_n) *converges to a in metric space* (X, d).

Proof. Since for any $x \in X$, $\rho(x,x) = 0$, then from definition of d , we have $d(x,x) = 0$. For all $x, y \in X$ with $x \neq y$, we will show that $d(x,y) > 0$. Suppose $d(x,y) = 0$. By using definition of the infimum, then for any $\varepsilon > 0$, there exists a positive integer M such that

$$\sum_{j=1}^{M-1} b^j \rho(a_j, a_{j+1}) < \varepsilon.$$

Since f no-decreasing, then we have

$$f\left(\sum_{j=1}^{M-1} b^j \rho(a_j, a_{j+1})\right) \leq f(\varepsilon). \tag{16}$$

Since $\rho(x,y) > 0$, then by using of B3, we have

$$f\big(\rho(x,y)\big) < f\left(\sum_{j=1}^{M-1} b^j \rho(a_j, a_{j+1})\right) + K. \tag{17}$$

From (16) and (17) we obtain

$$f\big(\rho(x,y)\big) < f(\varepsilon) + K. \tag{18}$$

However, from the logarithmic-like property of f, if $\varepsilon \to 0$, then we have $f(\varepsilon) \to -\infty$. It implies that $f(\varepsilon) + K \to -\infty$. It is a contradiction of (18). Thus, it concludes if $x \neq y$, $d(x,y) > 0$.

For condition 2 of metric, from definition of d, it is clear that $d(x,y) = d(y,x)$. To check that condition 3 metric, it can be shown as follows: Let $(a_n) \subset X$, $a_1 = x, a_2, \ldots, a_s = y$ and $a_s = y, a_{s+1}, \ldots, a_N = z$. From the definition of d, then for every $\varepsilon > 0$, there exists $a_1 = x, a_2, \ldots, a_s = y$ and $a_s = y, a_{s+1}, \ldots, a_N = z$ such that

$$\sum_{j=1}^{s-1} b^j \rho(a_j, a_{j+1}) < d(x,y) + \varepsilon,$$

and

$$\sum_{j=s}^{N-1} b^j \rho(a_j, a_{j+1}) < d(y,z) + \varepsilon.$$

So we have

$$d(x,z) = \inf_{N\in\mathbb{N}} \left\{ \sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1}) \mid (a_n) \subset X, a_1 = x, a_N = z, N \geq 2 \right\}$$

$$\leq \sum_{j=1}^{N-1} b^j \rho(a_j, a_{j+1}) = \sum_{j=1}^{s-1} b^j \rho(a_j, a_{j+1}) + \sum_{j=s}^{N-1} b^j \rho(a_j, a_{j+1})$$

$$< d(x,y) + d(y,z) + 2\varepsilon. \quad (19)$$

Since $\varepsilon > 0$ is arbitrary, then from (19) we get

$$d(x,z) \leq \ d(x,y) + d(y,z).$$

Thus, (X, ρ) is a metrizable by the metric d.

Next, it will be shown (a_n) converges to a in (X, ρ) if and only if (a_n) converges to a in (X, d). Let (a_n) converges to a in (X, ρ), It means for every $\varepsilon > 0$, there is a positive integer N, such that for any $n \geq N$, $\rho(a_n, a) < \varepsilon$. So, from the definition of d, we have

$$d(a_n, a) \leq \rho(a_n, a) < \varepsilon.$$

Thus, (a_n) converges to a in (X, d). Converse, let (a_n) converges to a in (X, d). It means for every $\varepsilon > 0$, there is a positive integer N, such that for any $n \geq N$, $d(a_n, a) < \varepsilon$. From the definition of d, for $\delta > 0$ then there will be a positive integer M such that

$$\sum_{j=1}^{M-1} b^j \rho(x_j, x_{j+1}) < \delta.$$

So, we have

$$f\left(\sum_{j=1}^{M-1} b^j \rho(x_j, x_{j+1}))\right) < f(\varepsilon) - K. \quad (20)$$

From $\rho(a_n, a) > 0$, and of B3 we have

$$f(\rho(a_n, a)) \leq f\left(\sum_{j=1}^{M-1} b^j \rho(x_j, x_{j+1}))\right) + K, \tag{21}$$

where $x_1 = a_n, x_M = a, M \geq 2$. From (20) and (21), we obtain

$$f(\rho(a_n, a)) < f(\varepsilon).$$

Thus, we have $\rho(a_n, a) < \varepsilon,$ for any $n \geq N$. It concludes that (a_n) converges to a in (X, ρ).

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